Proof of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{1}{0!}+\frac{1}{1!}+\cdots+\frac{1}{n!}\right)$

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The Euler's number

$$e = 2.718\cdots$$

can be defined in two ways:

1) $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

2) $e = \lim_{n\to\infty}\left(\frac{1}{0!}+\frac{1}{1!}+\cdots+\frac{1}{n!}\right)$

We will prove that the two limits exists, and they are equal to each other. Throughout this note, we denote the two sequences as

$$a_n := \left(1+\frac{1}{n}\right)^n$$

and

$$b_n := \left(1+\frac{1}{n+1}\right)^{n+1}$$

Step I. a_n is increasing

Proof. Expand

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &= 1+\sum_{k=1}^{n}\binom{n}{k}\frac{1}{n^k} \end{aligned}$$

and

$$\begin{aligned} a_{n+1} &= \left(1+\frac{1}{n+1}\right)^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(n+1)^k} \\ &= 1 + \frac{1}{(n+1)^{n+1}} + \sum_{k=1}^{n} \frac{(n+1)!}{k!(n+1-k)!} \cdot \frac{1}{(n+1)^k} \\ &= 1 + \frac{1}{(n+1)^{n+1}} + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{1}{(n+1-k)(n+1)^{k-1}} \\ &= 1 + \frac{1}{(n+1)^{n+1}} + \sum_{k=1}^{n} \binom{n}{k} \cdot \frac{1}{(n+1)^k - k(n+1)^{k-1}} \end{aligned}$$

Then

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{(n+1)^{n+1}} + \sum_{k=1}^{n} \binom{n}{k} \cdot \left(\frac{1}{(n+1)^k - k(n+1)^{k-1}} - \frac{1}{n^k} \right) \\ &= \frac{1}{(n+1)^{n+1}} + \sum_{k=1}^{n} \binom{n}{k} \cdot \frac{n^k - (n+1)^k + k(n+1)^{k-1}}{n^k(n+1-k)(n+1)^{k-1}} \end{aligned}$$

We want to show the above expression is > 0, only need to show that the numerators

$$\alpha_k := n^k - (n+1)^k + k(n+1)^{k-1} \geq 0$$

for $k = 1, 2, \ldots, n$. We proof this by induction.

(1) If $k = 1$,

$$n^k - (n+1)^k + k(n+1)^{k-1} = n - (n+1) + 1 = 0,$$

thus $\alpha_1 \geq 0$ is true.

(2) Suppose $\alpha_k \geq 0$ is true, where $k \geq 1$, notice the following

$$\begin{aligned} 0 \leq (n+1) \cdot \alpha_k &= (n+1) \cdot (n^k - (n+1)^k + k(n+1)^{k-1}) \\ &= n^{k+1} + n^k - (n+1)^{k+1} + k(n+1)^k \\ &= n^{k+1} - (n+1)^{k+1} + (k+1)(n+1)^k + n^k - (n+1)^k \\ &= \alpha_{k+1} + n^k - (n+1)^k \\ &\leq \alpha_{k+1} \end{aligned}$$

therefore $\alpha_{k+1} \geq 0$. This finishes the proof. □

Step II. a_n is bounded, therefore $\lim_{n\to\infty} a_n$ exists

We will prove the following:

$$a_n \leq b_n < 3$$

II.1. $b_n < 3$

Proof. Notice for any $n \geq 1$,

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n \geq 1 \cdot \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{(n-1) \text{ terms}} = 2^{n-1}$$

thus

$$\begin{aligned} b_n &= \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 3 - \frac{1}{2^{n-1}} < 3 \end{aligned}$$

□

II.2. $a_n \leq b_n$

Proof. This is straightforward:

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} = b_n \end{aligned}$$

□

Step III. $b_n \leq \lim_{m\to\infty} a_m$

Once this is proved, combining Step II we have

$$a_n \leq b_n \leq \lim_{m\to\infty} a_m$$

then let $n \to \infty$, we get $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

Proof. Use the expansion in Step II.2

$$\begin{aligned} a_m &= \sum_{k=0}^{m} \frac{1}{k!} \cdot \frac{m(m-1)(m-2)\cdots(m-k+1)}{m^k} \\ &= \sum_{k=0}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) \\ &= \sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) + \sum_{k=n+1}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) \end{aligned}$$

Now let $m \to \infty$ on both sides

$$\begin{aligned}
\lim_{m\to\infty} a_m &= \lim_{m\to\infty} \sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) \\
&\quad + \lim_{m\to\infty} \sum_{k=n+1}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) \\
&= \sum_{k=0}^{n} \frac{1}{k!} + \lim_{m\to\infty} \sum_{k=n+1}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) \\
&= b_n + \lim_{m\to\infty} \sum_{k=n+1}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) \\
&\geq b_n
\end{aligned}$$

$\square$